Supplementary Material

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1 Proof

1.1 Proof of Proposition 4.1

Proof. Given the noiseless physical observation model $\mathbf{y} = \mathbf{A}\mathbf{x}_0$, it necessarily holds that $\mathbf{y} \in \text{Range}(\mathbf{A})$. By the properties of the Moore-Penrose pseudoinverse, the matrix $\mathbf{A}\mathbf{A}^\dagger$ is the orthogonal projection matrix onto the range of the operator $\mathbf{A}$, denoted as $\text{Range}(\mathbf{A})$. Thus, for any $\mathbf{y} \in \text{Range}(\mathbf{A})$, we have:

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{y} = \mathbf{y}$$

Meanwhile, based on the definition of the null-space projection matrix $P_N = \mathbf{I} - \mathbf{A}^\dagger\mathbf{A}$ and the pseudoinverse identity $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$, left-multiplying by the degradation operator $\mathbf{A}$ yields:

$$\mathbf{A}P_N = \mathbf{A}(\mathbf{I} - \mathbf{A}^\dagger\mathbf{A}) = \mathbf{A} - \mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A} - \mathbf{A} = \mathbf{0}$$

Substituting these operator properties into the modulated estimation formulation of UCPM, $\hat{\mathbf{x}} = \mathbf{A}^\dagger\mathbf{y} + P_N\mathbf{x}_{Prior}$, and left-multiplying both sides by $\mathbf{A}$, we obtain:

$$\begin{aligned}\mathbf{A}\hat{\mathbf{x}} &= \mathbf{A}\left(\mathbf{A}^\dagger\mathbf{y} + P_N\mathbf{x}_{Prior}\right)\\ &= \mathbf{A}\mathbf{A}^\dagger\mathbf{y} + \mathbf{A}P_N\mathbf{x}_{Prior}\\ &= \mathbf{y} + \mathbf{0}\cdot\mathbf{x}_{Prior}\\ &= \mathbf{y}\end{aligned}$$

This demonstrates that the prior modulation variable $\mathbf{x}_{Prior}$, induced by epistemic uncertainty, is constrained within the null space of the operator $\mathbf{A}$ by P_N. Consequently, it does not interfere with the existing physical observations. Thus, the data fidelity residual is:

$$||\mathbf{y} - \mathbf{A}\hat{\mathbf{x}}||_2^2 = ||\mathbf{y} - \mathbf{y}||_2^2 = 0$$

This completes the proof. □

1.2 Proof of Proposition 4.2

Proof. Consider the probability flow ODE for the reverse diffusion process: $d\mathbf{x}/dt = f(\mathbf{x}, t)$. By assumption, the drift field f satisfies L-Lipschitz continuity in the state space:

$$||f(\mathbf{x}_1, t) - f(\mathbf{x}_2, t)||_2 \le L||\mathbf{x}_1 - \mathbf{x}_2||_2, \quad \forall \mathbf{x}_1, \mathbf{x}_2$$

Let t_k and t_{k+1} be adjacent time steps, with a step size of $\Delta t_k = t_{k+1} - t_k$. The single-step exact evolution of the true solution $\mathbf{x}(t)$ and the single-step update of the SATP numerical solution $\tilde{\mathbf{x}}$ satisfy, respectively:

$$\mathbf{x}(t_{k+1}) = \mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} f(\mathbf{x}(s), s)ds$$

$$\tilde{\mathbf{x}}_{k+1} = \tilde{\mathbf{x}}_k + \Delta t_k f(\tilde{\mathbf{x}}_k, t_k)$$

Let $e_k = ||\mathbf{x}(t_k) - \tilde{\mathbf{x}}_k||_2$ denote the global evolution error at time t_k. Following numerical ODE theory, we define the local truncation error at the k-th step, $\boldsymbol{\tau}_k$, as:

$$\boldsymbol{\tau}_k = \mathbf{x}(t_{k+1}) - [\mathbf{x}(t_k) + \Delta t_k f(\mathbf{x}(t_k), t_k)]$$

The SATP mechanism strictly bounds this local truncation error such that $||\boldsymbol{\tau}_k||_2 \le CB\Delta t_k$. Combining the definition of $\boldsymbol{\tau}_k$ with the triangle inequality, we can expand and bound e_{k+1} as follows:

$$\begin{aligned}e_{k+1} &= ||\mathbf{x}(t_{k+1}) - \tilde{\mathbf{x}}_{k+1}||_2\\ &= ||\mathbf{x}(t_k) + \Delta t_k f(\mathbf{x}(t_k), t_k) + \boldsymbol{\tau}_k - \tilde{\mathbf{x}}_k - \Delta t_k f(\tilde{\mathbf{x}}_k, t_k)||_2\\ &\le ||\mathbf{x}(t_k) - \tilde{\mathbf{x}}_k||_2 + \Delta t_k||f(\mathbf{x}(t_k), t_k) - f(\tilde{\mathbf{x}}_k, t_k)||_2 + ||\boldsymbol{\tau}_k||_2\end{aligned}$$

Applying the Lipschitz continuity condition and the truncation error bound yields:

$$e_{k+1} \le e_k + L\Delta t_k e_k + CB\Delta t_k = (1 + L\Delta t_k)e_k + CB\Delta t_k$$

Using the inequality $1 + x \le e^x$, this discrete error recurrence relation can be further bounded by:

$$e_{k+1} \le e^{L\Delta t_k}e_k + CB\Delta t_k$$

Assuming perfect sampling of the initial prior state ($e_0 = 0$), we can unroll the recurrence relation from $k = 0$ up to the final time t_N:

$$\begin{aligned}e_N &\le \sum_{j=0}^{N-1} e^{L\sum_{i=j+1}^{N-1}\Delta t_i} CB\Delta t_j\\ &= CB\sum_{j=0}^{N-1} e^{L(t_N - t_{j+1})}\Delta t_j\end{aligned}$$

Since the integrand $e^{L(t_N - s)}$ is monotonically decreasing on $s \in [t_j, t_{j+1}]$, the right-hand side can be strictly upper-bounded by the corresponding definite integral:

$$e_N \le CB\int_0^{t_N} e^{L(t_N - s)}ds$$

Evaluating this integral analytically gives:

$$e_N \le \frac{CB}{L}\left[-e^{L(t_N - s)}\right]_0^{t_N} = \frac{CB}{L}(e^{Lt_N} - 1)$$

Given that the total generation time is $t_N \le T$, and the exponential function is monotonically increasing, we have $e_N \le \frac{CB}{L}(e^{LT} - 1)$. Meanwhile, note that the terminal evolution error e_N is exactly the ℓ_2-norm deviation between the continuous exact solution $\mathbf{x}^{Dense}$ and the SATP discrete solution $\mathbf{x}^{LEADer}$, i.e., $e_N = ||\mathbf{x}^{Dense} - \mathbf{x}^{LEADer}||_2$. Substituting this into the above inequality, we obtain the final global error bound:

$$||\mathbf{x}^{Dense} - \mathbf{x}^{LEADer}||_2 \le \frac{C\cdot B}{L}(e^{LT} - 1)$$

This completes the proof. □